Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 4, November 2003, pp. 453–459.
Printed in India

Topologically left invariant means on semigroup algebras

ALI GHAFFARI

Department of Mathematics, Semnan University, Semnan, Iran
E-mail: ghaffari1380@yahoo.com

MS received 22 February 2005; revised 17 July 2005

Abstract. Let $M(S)$ be the Banach algebra of all bounded regular Borel measures on a locally compact Hausdorff semitopological semigroup S with variation norm and convolution as multiplication. We obtain necessary and sufficient conditions for $M(S)^*$ to have a topologically left invariant mean.

Keywords. Banach algebras; locally compact semigroup; topologically left invariant mean; fixed point.

1. Introduction

Let S be a locally compact Hausdorff semitopological semigroup with convolution measure algebra $M(S)$ and probability measures $M_0(S)$. We know that $M(S)$ is a Banach algebra with total variation norm and convolution. The first Arens multiplication on $M(S)^*$ is defined in three steps as follows.

For μ,ν in $M(S)$, f in $M(S)^*$ and F,G in $M(S)^{**}$, the elements $f\mu, Ff$ of $M(S)^*$ and GF of $M(S)^{**}$ are defined by

$$\langle f\mu,\nu\rangle = \langle f,\mu*\nu\rangle,\quad \langle Ff,\mu\rangle = \langle F,f\mu\rangle,\quad \langle GF,f\rangle = \langle G,Ff\rangle.$$

Denote by 1 the element in $M(S)^*$ such that $\langle 1,\mu\rangle = \mu(S), \mu\in M(S)$. A linear functional $M\in M(S)^{**}$ is called a *mean* if $\langle M,f\rangle \geq 0$ whenever $f\geq 0$ and $\langle M,1\rangle = 1$. Each probability measure $\mu\in M_0(S)$ is a mean. An application by the Hahn–Banach theorem shows that $M_0(S)$ is weak* dense in the set of means on $M(S)^*$. A mean M is *topological left invariant* if $\langle M, f\mu\rangle = \langle M,f\rangle$ for any $\mu\in M_0(S)$ and $f\in M(S)^*$. We shall follow Ghaffari [7] and Wong [14,15] for definitions and terminologies not explained here. We know that topologically left invariant mean on $M(S)^*$ have been studied by Riazi and Wong in [11] and by Wong in [14,15]. They also went further and for several subspaces X of $M(S)^*$, have obtained a number of interesting and nice results.

The existence of topologically left invariant means and left invariant means for groups was investigated widely by Paterson [9] and Pier [10]. Other important studies on amenable semigroups are those of Argabright [1], Day [4], Lau [6], and Mitchell [8]. For further studies and complementary historical comments see [3,9,10].

Let $M_0(S)$ have a measure ν such that the map $s\mapsto \delta_s*\nu$ from S into $M(S)$ is continuous. The author recently proved that the following conditions are equivalent:

(a) $M(S)^*$ has a topologically left invariant mean;
(b) there is a net (μ_α) in $M_0(S)$ such that for every compact subset K of S, $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly over all μ in $M_0(S)$ which are supported in K.

In this paper, we obtain a necessary and sufficient condition for $M(S)^*$ to have a topologically left invariant mean.

2. Main results

Throughout the paper, S is a locally compact Hausdorff semitopological semigroup. We say that S is *semifoundation* if there is a measure $\nu \in M_0(S)$ such that the map $x \mapsto \delta_x * \nu$ from S into $M(S)$ is continuous. It is clear that every foundation semigroup is also a semifoundation semigroup (for more on foundation semigroups, the reader is referred to [2] and [5]). We recall that a mean M is *left invariant* if $\langle M, f\delta_x\rangle = \langle M, f\rangle$ for any $x \in S$ and $f \in M(S)^*$. Obviously, a topologically left invariant mean on $M(S)^*$ is also a left invariant mean on $M(S)^*$.

PROPOSITION 1.

Let S be a semifoundation semigroup. Choose $\nu \in M_0(S)$ such that the map $x \mapsto \delta_x * \nu$ from S into $M(S)$ is continuous. If $M \in M(S)^{**}$ is a left invariant mean on $M(S)^*$ and $\langle M, f\nu\rangle = \langle M, f\rangle$ for each $f \in M(S)^*$, then M is a topologically left invariant mean on $M(S)^*$.

Note that this is Proposition 22.2 of [10], which was proved for groups. However, our proof is completely different.

Proof. Suppose that $f \in M(S)^*$ and $\mu \in M_0(S)$. For every $x \in S$, we can write

$$\langle M, f\delta_x * \nu\rangle = \langle M, f\delta_x\rangle = \langle M, f\rangle.$$

It follows that

$$\int \langle Mf, \delta_x * \nu\rangle \mathrm{d}\mu(x) = \langle M, f\rangle. \tag{1}$$

Since $x \mapsto \delta_x * \nu$ is continuous, by Theorem 3.27 in [12], it is easy to see that

$$\int \langle Mf, \delta_x * \nu\rangle \mathrm{d}\mu(x) = \langle Mf, \mu * \nu\rangle = \langle M, f\mu * \nu\rangle. \tag{2}$$

Hence, using (1) and (2), $\langle M, f\mu * \nu\rangle = \langle M, f\rangle$. By hypothesis,

$$\langle M, f\mu * \nu\rangle = \langle M, (f\mu)\nu\rangle = \langle M, f\mu\rangle.$$

Consequently $\langle M, f\rangle = \langle M, f\mu\rangle$, i.e., M is a topologically left invariant mean on $M(S)^*$. □

Our next result gives an important property that characterizes topologically left amenability of $M(S)^*$.

Theorem 1. *Let S be semifoundation semigroup with identity. Then the following statements are equivalent*:

(1) *$M(S)^*$ has a topologically left invariant mean;*
(2) *for all $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in M(S)$,*

$$\inf\{\sup\{\|\mu_i * \mu\|; 1 \le i \le n\}, \mu \in M_0(S)\} \le \sup\{|\mu_i(S)|; 1 \le i \le n\}.$$

Proof. Let $M(S)^*$ have a topologically left invariant mean. Let $\mu_1, \ldots, \mu_n \in M(S)$, $\varepsilon > 0$ and put $\delta = \varepsilon(2 + \sup\{\|\mu_i\|; 1 \le i \le n\})^{-1}$. There exists a compact subset K in S such that $|\mu_i|(S \backslash K) < \delta$ whenever $i = 1, \ldots, n$. By Theorem 2.2 in [7], there exists a measure μ in $M_0(S)$ such that the map $x \mapsto \delta_x * \mu$ from S into $M(S)$ is continuous and $\|\delta_x * \mu - \mu\| < \delta$ for any $x \in K$. Thus, for every $i = 1, \ldots, n$ and $f \in M(S)^*$, by Theorem 3.27 in [12], we can write

$$\begin{aligned}
|\langle f, \mu_i * \mu - \mu_i(S)\mu\rangle| &= \left|\int \langle f, \delta_x * \mu\rangle \mathrm{d}\mu_i(x) - \mu_i(S)\langle f, \mu\rangle\right| \\
&= \left|\int (\langle f, \delta_x * \mu\rangle - \langle f, \mu\rangle) \mathrm{d}\mu_i(x)\right| \\
&= \left|\int \langle f, \delta_x * \mu - \mu\rangle \mathrm{d}\mu_i(x)\right| \\
&= \left|\int_{S \backslash K} \langle f, \delta_x * \mu - \mu\rangle \mathrm{d}\mu_i(x)\right. \\
&\qquad \left. + \int_K \langle f, \delta_x * \mu - \mu\rangle \mathrm{d}\mu_i(x)\right| \\
&\le 2|\mu_i|(S \backslash K)\|f\| + \delta\|f\||\mu_i|(K) \\
&\le 2\delta\|f\| + \delta\|f\|\|\mu_i\| \\
&= \delta\|f\|(2 + \|\mu_i\|) \le \varepsilon\|f\|.
\end{aligned}$$

It follows that $\|\mu_i * \mu - \mu_i(S)\mu\| < \varepsilon$ whenever $i = 1, \ldots, n$. Consequently

$$\sup\{\|\mu_i * \mu\|; 1 \le i \le n\} \le \sup\{|\mu_i(S)|; 1 \le i \le n\} + \varepsilon.$$

Therefore

$$\inf\{\sup\{\|\mu_i * \mu\|; 1 \le i \le n\}; \mu \in M_0(S)\} \le \sup\{|\mu_i(S)|; 1 \le i \le n\}.$$

Conversely let $\mu_1, \ldots, \mu_n \in M_0(S)$ and $\varepsilon > 0$. For any $i = 1, \ldots, n$, consider $\nu_i = \mu_i - \delta_e$. We have $\nu_i(S) = 0$ whenever $i = 1, \ldots, n$. By assumption,

$$\inf\{\sup\{\|\nu_i * \mu\|; 1 \le i \le n\}; \mu \in M_0(S)\} = 0.$$

Thus there exists $\mu \in M_0(S)$ such that

$$\sup\{\|\nu_i * \mu\|; 1 \le i \le n\} < \varepsilon,$$

i.e., for every $i = 1, \ldots, n$, $\|\mu_i * \mu - \mu\| < \varepsilon$. By Theorem 2.2 in [7], $M(S)^*$ has a topologically left invariant mean. □

Let V be a locally convex Hausdorff topological vector space and let Z be a compact convex subset of V. The pair $(M_0(S),Z)$ is called a *semiflow*, if;

(1) There exists a map ρ: $M(S)\times V\to V$ such that for every $z\in Z$, the map $\rho(-,z)$: $M(S)\to V$ is continuous and linear ($M(S)$ has the topology $\sigma(M(S), M(S)^*)$);
(2) $\rho(M_0(S),Z)\subseteq Z$;
(3) For any $\mu,\nu\in M(S)$ and $z\in Z, \rho(\mu,\rho(\nu,z))=\rho(\mu*\nu,z)$.

We remind the reader of our notation conventions:

$$\mu z=\rho(\mu,z),\quad \mu\in M(S),z\in Z.$$

Theorem 2. *Let S be a semitopological semigroup. The following statements are equivalent*:

(1) *$M(S)^*$ has a topologically left invariant mean*;
(2) *for every $f\in M(S)^*$, there exists a mean M such that $\langle M,f\mu\rangle=\langle M,f\nu\rangle$ for any μ,ν in $M_0(S)$*;
(3) *for any semiflow $(M_0(S),Z)$, there is some $z\in Z$ such that $\mu z=z$ for all $\mu\in M_0(S)$.*

Proof. (1) implies (2) is easy.

Now, assume that (2) holds. We will show that $M(S)^*$ has a topologically left invariant mean. To each $f\in M(S)^*$, we associate the non-void subset

$$\Omega_f=\{M\in\Omega;\langle M,f\mu\rangle=\langle M,f\nu\rangle\quad\text{for all }\mu,\nu\in M_0(S)\},$$

(Ω is the convex set of all means on $M(S)^*$.) The sets Ω_f are obviously weak* compact. We shall show that the family $\{\Omega_f; f\in M(S)^*\}$ has the finite intersection property. Since Ω is weak* compact, it will follow that

$$\bigcap\{\Omega_f; f\in M(S)^*\}\neq\emptyset;$$

and if M is any member of this intersection, then M^2 is a topologically left invariant mean on $M(S)^*$.

We proceed by induction. By hypothesis, $\Omega_f\neq\emptyset$ for each $f\in M(S)^*$. Let $n\in\mathbb{N}$, $f_1,\ldots,f_n\in M(S)^*$ and assume that $\cap_{i=1}^{n-1}\Omega_{f_i}\neq\emptyset$. If M_1 is a member of this intersection and if $M_2\in\Omega_{M_1f_n}$, then for every μ,ν in $M_0(S)$ we have

$$\langle M_2M_1, f_n\mu\rangle=\langle M_2,M_1f_n\mu\rangle=\langle M_2,M_1f_n\nu\rangle=\langle M_2M_1, f_n\nu\rangle$$

and, for $i=1,\ldots,n-1$,

$$\begin{aligned}\langle M_2M_1, f_i\mu\rangle&=\langle M_2,M_1f_i\mu\rangle=\lim_\alpha\langle\mu_\alpha,M_1f_i\mu\rangle\\&=\lim_\alpha\langle M_1f_i\mu,\mu_\alpha\rangle=\lim_\alpha\langle M_1,(f_i\mu)\mu_\alpha\rangle\\&=\lim_\alpha\langle M_1, f_i\mu*\mu_\alpha\rangle=\lim_\alpha\langle M_1, f_i\nu*\mu_\alpha\rangle\\&=\lim_\alpha\langle M_1,(f_i\nu)\mu_\alpha\rangle=\lim_\alpha\langle\mu_\alpha,M_1f_i\nu\rangle\\&=\langle M_2M_1, f_i\nu\rangle.\end{aligned}$$

(Recall that $M_0(S)$ is weak* dense in Ω, and so there is a net $\{\mu_\alpha\}$ in $M_0(S)$ such that $\mu_\alpha \to M_2$ in the weak* topology.) Hence $M_2M_1 \in \cap_{i=1}^n \Omega_{f_i}$. Thus $\{\Omega_f; f \in M(S)^*\}$ has the finite intersection property, as required. So (1) is equivalent to (2).

To prove that (1) and (3) are equivalent, let $(M_0(S), Z)$ be a semiflow on a compact convex subset Z of a locally convex Hasudorff topological vector space V. If $f \in V^*$ and $z \in Z$, we consider the mapping f^z: $M(S) \to \mathbb{C}$ given by $\langle f^z, \mu\rangle = \langle f, \mu z\rangle$. It is easy to see that $f^z \in M(S)^*$. Let Ω be the convex set of all means on $M(S)^*$. For $M \in \Omega$, we can define $T(M)$: $V^* \to \mathbb{C}$ given by $\langle T(M), g\rangle = \langle M, g^z\rangle$ $(g \in V^*)$. One easily notes that $T(M)$ is linear. Now, we embed V into the algebraic dual $V^{*\prime}$ of V^* with the topology $\sigma(V^{*\prime}, V^*)$. Since Z is compact in V, it is closed in $V^{*\prime}$. On the other hand, for every $h \in V^*$ and $\mu \in M_0(S)$, we have

$$\langle T(\mu), h\rangle = \langle \mu, h^z\rangle = \langle h, \mu z\rangle = \langle \mu z, h\rangle.$$

It follows that the $M_0(S)$-invariance of Z implies that $T(\mu) \in Z$. Since $M_0(S)$ is weak*-dense in Ω and Z is closed in $V^{*\prime}$, we conclude that $T(M) \in Z$ for every $M \in \Omega$. If $\mu \in M_0(S)$, we consider λ_μ: $Z \to Z$ by $\lambda_\mu(z) = \mu z (z \in Z)$. Now let M be a topologically left invariant mean on $M(S)^*$. For every $h \in V^*$ and $\mu \in M_0(S)$, we have

$$\begin{aligned}\langle \mu T(M), h\rangle &= \langle T(M), h \circ \lambda_\mu\rangle = \langle M, (h \circ \lambda_\mu)^z\rangle \\ &= \langle M, h^z \mu\rangle = \langle M, h^z\rangle \\ &= \langle T(M), h\rangle.\end{aligned}$$

So $\mu T(M) = T(M)$ for every $\mu \in M_0(S)$, i.e., $T(M)$ is a fixed point under the action of $M_0(S)$.

To prove the converse, we know that the set Ω is convex and weak*-compact in $M(S)^{**}$. We define the semiflow $(M_0(S), \Omega)$ by putting $\rho(\mu, F) = \mu F$ for $\mu \in M(S)$ and $F \in M(S)^{**}$. By hypothesis, there exists $M \in \Omega$ that is fixed under the action of $M_0(S)$, that is $\mu M = M$ for every $\mu \in M_0(S)$. It follows that M is a topologically left invariant mean on $M(S)^*$. This completes our proof. □

A *right action* of $M(S)$ on $M(S)^*$ is a map T: $M(S) \times M(S)^* \to M(S)^*$ (denoted by $(\mu, f) \mapsto T_\mu(f), \mu \in M(S)$ and $f \in M(S)^*$) such that

(1) $(\mu, f) \mapsto T_\mu(f)$ is bilinear and $T_{\mu * \nu} = T_\nu \circ T_\mu$ for any $\mu, \nu \in M(S)$,

(2) T_μ: $M(S)^* \to M(S)^*$ is a positive linear operator and $T_\mu(1) = 1$ for any $\mu \in M_0(S)$.

Let X be a linear subspace of $M(S)^*$ with $1 \in X$. We say that $M \in X^*$ is a *mean* on X if $\langle M, f\rangle \geq 0$ if $f \geq 0$ and $\langle M, 1\rangle = 1$. A mean M is $M_0(S)$-invariant under the right action T if $\langle M, T_\mu(f)\rangle = \langle M, f\rangle$ for any $\mu \in M_0(S)$ and $f \in X$. We say that X is $M_0(S)$-invariant under the right action T if $T_\mu(X) \subseteq X$ for any $\mu \in M_0(S)$.

Theorem 3. *Let S be a semitopological semigroup. The following satatements are equivalent*:

(1) *$M(S)^*$ has a topologically left invariant mean*;

(2) *for any separately continuous right action T: $M(S) \times M(S)^* \to M(S)^*$ of $M(S)$ on $M(S)^*$ ($M(S)$ has the topology $\sigma(M(S), M(S)^*)$ and $M(S)^*$ has the weak topology) and any $M_0(S)$-invariant subspace X of $M(S)^*$ containing 1, any $M_0(S)$-invariant mean M on X can be extended to a $M_0(S)$-invariant mean $\mathscr{M}$ on $M(S)^*$.*

Proof. Let $M(S)^*$ have a topologically left invariant mean, and let

$$T\colon M(S)\times M(S)^*\to M(S)^*$$

be a separately continuous right action of $M(S)$ on $M(S)^*$ and M be a mean on $M_0(S)$-invariant subspace X of $M(S)^*$. Let

$$Z=\{\mathscr{M}\in\mathscr{M}(\mathscr{S})^{**};\mathscr{M}\text{ is a mean on }\mathscr{M}(\mathscr{S})^{**}\text{ and extends }\mathscr{M}\}.$$

By the Hahn–Banach theorem, $Z\neq\emptyset$. It is easy to see that Z is a weak* closed convex subset of the unit ball in $M(S)^{**}$, and is therefore weak* compact. Define $\rho\colon M(S)\times M(S)^{**}\to M(S)^{**}$ by $\rho(\mu,F)=T_\mu^*(F),\mu\in M(S)$, $F\in M(S)^{**}$. Notice that, since $T\colon M(S)\times M(S)^*\to M(S)^*$ is a separately continuous right action of $M(S)$ on $M(S)^*$, it is clear that

$$\rho(-,F)\colon M(S)\to M(S)^{**}$$

is continuous for any $F\in M(S)^{**}(M(S)$ has the topology $\sigma(M(S),M(S)^*)$ and $M(S)^{**}$ has the topology $\sigma(M(S)^{**},M(S)^*))$. On the other hand, it is clear that each $\rho(-,F)\colon M(S)\to M(S)^{**}$ is linear since $T\colon M(S)\times M(S)^*\to M(S)^*$ is bilinear. Let $\mathscr{M}\in\mathscr{Z}$ and $\mu\in M_0(S)$. Since $T_\mu\colon M(S)^*\to M(S)^*$ is positive linear and $T_\mu(1)=1$, so $T_\mu^*(\mathscr{M})$ is a mean on $M(S)^*$. Now, let $f\in X$ we have

$$\langle T_\mu^*(\mathscr{M}),\{\rangle=\langle\mathscr{M},\mathscr{T}_\mu(\{)\rangle=\langle\mathscr{M},\mathscr{T}_\mu(\{)\rangle=\langle\mathscr{M},\{\rangle.$$

This shows that $\rho(\mu,\mathscr{M})=\mathscr{T}_\mu^*(\mathscr{M})\in\mathscr{Z}$, i.e., $\rho(M_0(S),Z)\subseteq Z$. Let $\mu,\nu\in M(S)$ and $\mathscr{M}\in\mathscr{Z}$. Since $T\colon M(S)\times M(S)^*\to M(S)^*$ is an anti-homomorphism of $M(S)$ into the algebra of linear operators in $M(S)^*$, therefore

$$\begin{aligned}\langle\rho(\mu,\rho(\nu,\mathscr{M})),\{\rangle&=\langle T_\mu^*(\rho(\nu,\mathscr{M})),\{\rangle=\langle\mathscr{T}_\mu^*(\mathscr{T}_\nu^*(\mathscr{M})),\{\rangle\\&=\langle T_\nu^*(\mathscr{M}),\mathscr{T}_\sqcap(\{)\rangle=\langle\mathscr{M},\mathscr{T}_\nu(\mathscr{T}_\mu(\{))\rangle\\&=\langle\mathscr{M},\mathscr{T}_{\mu*\nu}(\{)\rangle=\langle\mathscr{T}_{\mu*\nu}^*(\mathscr{M}),\{\rangle\\&=\langle\rho(\mu*\nu,\mathscr{M}),\{\rangle\end{aligned}$$

for any $f\in M(S)^*$. This shows that $\rho(\mu,\rho(\nu,\mathscr{M}))=\rho(\mu*\nu,\mathscr{M})$ for any μ,ν in $M(S)$ and $\mathscr{M}\in\mathscr{Z}$. As we saw above, the pair $(M_0(S),Z)$ is a semiflow. By Theorem 2, there is some $\mathscr{M}\in\mathscr{Z}$ such that $T_\mu^*(\mathscr{M})=\rho(\mu,\mathscr{M})=\mathscr{M}$ for each $\mu\in M_0(S)$. $\mathscr{M}$ is then the required extension of M.

Conversely, we define a right action $T\colon M(S)\times M(S)^*\to M(S)^*$ by putting $T_\mu(f)=f\mu$ for $\mu\in M(S)$ and $f\in M(S)^*$. We claim that it is separately continuous. If $\mu_\alpha\to\mu$ in the $\sigma(M(S),M(S)^*)$, then for any $F\in M(S)^{**}$, we have

$$\begin{aligned}\langle F,T_{\mu_\alpha}(f)\rangle&=\langle F,f\mu_\alpha\rangle=\langle Ff,\mu_\alpha\rangle\to\langle Ff,\mu\rangle\\&=\langle F,f\mu\rangle=\langle F,T_\mu(f)\rangle.\end{aligned}$$

On the other hand, it is easy to see that every $T_\mu\colon M(S)^*\to M(S)^*$ is continuous ($M(S)^*$ has the weak topology). Now choose X to be the constants and define $\langle M,\alpha\cdot 1\rangle=\alpha$, for any $\alpha\cdot 1\in X$. Then M is a mean on X satisfying $\langle M,T_\mu(f)\rangle=\langle M,f\rangle$ for any $\mu\in M_0(S)$ and $f\in X$. Any invariant extension $\mathscr{M}$ of M to $M(S)^*$ is necessarily a topologically left invariant mean on $M(S)^*$. This completes our proof. □

The above characterization of topologically left invariant mean on $M(S)^*$ is an analogue of Silverman's invariant extension property in [13].

References

[1] Argabright L N, Invariant means and fixed points. A sequel to Mitchells paper, *Trans. Am. Math. Soc.* **130** (1968) 127–130

[2] Baker A C and Baker J W, Algebras of measures on locally compact semigroups III, *J. London Math. Soc.* **4** (1972) 685–695

[3] Berglund J F, Junghenn H D and Milnes P, Analysis on semigroups, function spaces, compactifications, representations (1989) (New York)

[4] Day M M, Left thick to left lumpy a guided tour, *Pacific J. Math.* **101** (1982) 71–91

[5] Dzinotyiweyi H A M, The analogue of the group algebra for topological semigroups (1984) (Boston, London: Pitman)

[6] Lau A T, Amenability of semigroups, in: the analytical and topological theory of semigroups (eds) K H Hofman, J D Lawson and J S Pym (1999) (Berlin and New York: Walter de Gruyter) pp. 331–334

[7] Ghaffari A, Topologically left invariant mean on dual semigroup algebras, *Bull. Iranian Math. Soc.* **28** (2002) 69–75

[8] Mitchell T, Function algebras, means and fixed points, *Trans. Am. Math. Soc.* **130** (1968) 117–126

[9] Paterson A L T, Amenability, *Am. Math. Soc. Math. Survey and Monographs 29* (1988) (Rhode Island: Providence)

[10] Pier J P, Amenable locally compact groups (1984) (New York: John Wiley and Sons)

[11] Riazi A and Wong J C S, Characterizations of amenable locally compact semigroups, *Pacific J. Math.* **108** (1983) 479–496

[12] Rudin W, Functional analysis (1991) (New York: McGraw Hill)

[13] Silverman R J, Means on semigroups and the Hahn–Banach extension property, *Trans. Am. Math. Soc.* **83** (1956) 222–237

[14] Wong J C, A characterization of topological left thick subsets in locally compact left amenable semigroups, *Pacific J. Math.* **62** (1976) 295–303

[15] Wong J C, Fixed point theorems for measurable semigroups of operations, *Canad. J. Math.* **44** (1992) 652–664